CHAPTER 8

THEOREM OF MINIMUM POTENTIAL ENERGY, HAMILTON'S PRINCIPLE AND THEIR APPLICATIONS

8.1 Introduction

Many structures involve complicated shapes and numerous or unusual loads for which solutions of the governing differential equations and/or the boundary conditions are difficult or impossible. For instance, a rectangular plate with a hole somewhere, or a plate with discontinuous boundary conditions poses a major difficulty in finding an analytical solution.

For preliminary design and analysis one needs simplified, easy to use analyses analogous to those that have been presented earlier. However, for final design, quite often transverse shear deformation and thermal effects must be included. Thermal effects have been described in Chapter 4. Analytically they cause considerable difficulty, because with their inclusion few boundary conditions are homogeneous, hence separation of variables, used throughout the plate solutions to this point, cannot be utilized in a straightforward manner. Only through the laborious process of transformation of variables can the procedures discussed herein be used [1.1]. Therefore, energy principles are much more convenient for use in design and analyses of plate structures when thermal effects are present.

In solving plate problems it is seen that in order to obtain an analytical solution one must solve the differential equations and satisfy the boundary conditions; if that cannot be accomplished, there is no solution. With energy methods, one can always obtain a good approximate solution, no matter what the structural complexities, the loads or the boundary condition complications may be, using a little ingenuity.

In structural mechanics three energy principles are used: Minimum Potential Energy, Minimum Complementary Energy and Reissner's Variational Theorem [8.1]. The first two are discussed at length in Sokolnikoff [1.1] and many other references. The Reissner Variational Theorem, likewise, is widely referenced. In solid mechanics, Minimum Complementary Energy is rarely used, because it requires assuming functions that insure that the stresses satisfy boundary conditions and equilibrium. It is usually far easier to make assumptions about functions that can represent displacements.

Minimum Potential Energy is widely used in solutions to problems involving plate structures. In fact, the more complicated the loading, the more complicated the geometry and the more complicated the boundary conditions (e.g., discontinuous or concentrated boundary conditions), the more desirable it is to use Minimum Potential Energy to obtain an approximate solution, compared to attempting to solve the governing differential equations and to satisfy the boundary conditions exactly.

In addition, in many cases energy principles can be useful for eigenvalue problems such as in the buckling and vibration problems as shall be shown.

There are numerous books dealing with energy theorems and variational methods. One of the more recent is that by Mura and Koya [8.2].

8.2 Theorem of Minimum Potential Energy

For any generalized elastic body, the potential energy of that body can be written as follows:

$$V = \int_R W \mathrm{d}R - \int_{S_T} T_i u_i \mathrm{d}S - \int_R F_i u_i \mathrm{d}R$$

(8.1)

where

W = strain energy density function, defined in Equation (8.4) below
R = volume of the elastic body
T_i = ith component of the surface traction
u_i = ith component of the deformation
F_i = ith component of a body force
S_T = portion of the body surface over which tractions are prescribed

One sees that the first term on the right-hand side of Equation (8.1) is the strain energy of the elastic body. The second and third terms are the work done by the surface tractions; and the body forces, respectively. The Theorem of Minimum Potential Energy can be stated as described in [1.1]: "Of all the displacements satisfying compatibility and the prescribed boundary conditions, those that satisfy the equilibrium equations make the potential energy a minimum."

Mathematically, the operation is simply stated as,

$$\delta V = 0 \tag{8.2}$$

The lowercase delta is a mathematical operation known as a variation. Operationally, it is analogous to partial differentiation. To employ variational operations in structural mechanics, only the following three operations are usually needed (where y is any dependent variable):

$$\frac{\mathrm{d}(\delta y)}{\mathrm{d}x} = \delta\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right), \quad \delta\left(y^2\right) = 2y\delta y, \quad \int \delta y \mathrm{d}x = \delta \int y \mathrm{d}x \tag{8.3}$$

In Equation (8.1) the strain energy density function, W, is defined as follows in a Cartesian coordinate frame:

$$W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\sigma_x\varepsilon_x + \frac{1}{2}\sigma_y\varepsilon_y + \frac{1}{2}\sigma_z\varepsilon_z + \sigma_{xy}\varepsilon_{xy} + \sigma_{xz}\varepsilon_{xz} + \sigma_{yz}\varepsilon_{yz} \tag{8.4}$$

To utilize the Theorem of Minimum Potential Energy, the stress-strain relations for the elastic body are employed to change the stresses in Equation (8.4) to strains, and the strain-displacement relations are employed to change all strains to displacements. Thus, it is necessary for the analyst to select the proper stress-strain relations and strain-displacement relations for the problem being solved.

Although this text is dedicated to plate and panel structures, it is best to introduce the subject using isotropic monocoque beams, a much simpler structural component, to first illustrate the energy principles.

8.3 Analysis of a Beam In Bending Using the Theorem of Minimum Potential Energy

As the simplest example of the use of Minimum Potential Energy, consider a beam in bending, shown in Figure 8.1. In this section, Minimum Potential Energy methods are used to show that if one makes beam assumptions, one obtains the beam equation. However, the most useful employment of the Minimum Potential Energy Theorem is through making assumptions for the dependent variables (the deflection) and using the Theorem to obtain approximate solutions, as will be illustrated later.

From Figure 8.1 it is seen that the beam is of length L, in the x-direction, width b and height h. It is subjected to a lateral distributed load, $q(x)$ in the positive z-direction, in units of force per unit length. The modulus of elasticity of the isotropic beam material is E, and the stress-strain relation is simply

$$\sigma_x = \mathrm{E}\varepsilon_x \tag{8.5}$$

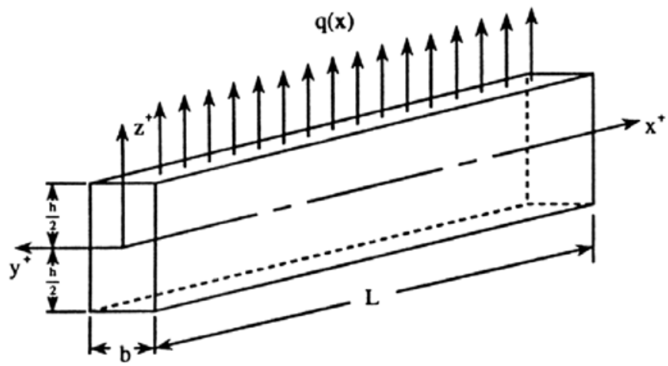


Figure 8.1. Beam in bending

The corresponding strain-displacement relation for a beam in bending only is, from (1.16), (2.1) and (2.27),

$$\varepsilon_x = \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} z \tag{8.6}$$

since in the bending of beams, $u = -z(\mathrm{d}w / \mathrm{d}x)$ only.

Looking at Equations (8.4) through (8.6) and remembering that in elementary beam theory

$$\sigma_y = \sigma_z = \sigma_{xy} = \varepsilon_{xz} = \varepsilon_{yz} = \sigma_{xy} = 0$$

then if the beam is subjected to bending only

$$W = \frac{1}{2}\sigma_x \varepsilon_x = \frac{1}{2}\mathrm{E}\varepsilon_x^2 = \frac{1}{2}\mathrm{E}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 z^2 \tag{8.7}$$

Therefore, the strain energy, U, which is the volume integral of the strain energy density function, W, is

$$U = \int_0^L \int_{-b/2}^{+b/2} \int_{-h/2}^{+h/2} \frac{1}{2}\mathrm{E}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 z^2 \mathrm{d}z\mathrm{d}y\mathrm{d}x = \frac{\mathrm{EI}}{2}\int_0^L \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x \tag{8.8}$$

where, $\mathrm{I} = bh^3/12$, the flexural stiffness for a beam of rectangular cross-section.

Similarly, from the surface traction work term in Equation (8.1) it is seen that

$$\int_{S_T} T_i u_i \mathrm{d}s = \int_0^L q(x)\, w(x)\mathrm{d}x$$

Equation (8.1) then becomes

$$V = \frac{\mathrm{EI}}{2}\int_0^L \left(\frac{\mathrm{d}w^2}{\mathrm{d}x^2}\right)^2 \mathrm{d}x - \int_0^L q(x)\, w(x)\mathrm{d}x \tag{8.9}$$

Following Equation (8.2) and remembering Equation (6.3) then

$$\delta V = 0 = \frac{\mathrm{EI}}{2}\int_0^L \delta\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x - \int_0^L q(x)\, \delta w(x)\mathrm{d}x \tag{8.10}$$

The variation δ can be included under the integral, because the order of variation and integration can be interchanged. Also, there is no variation of E, I or $q(x)$ because they are all specified quantities.

Integrating by parts the first term on the right-hand side of Equation (8.10).

$$
\begin{aligned}
\frac{\mathrm{EI}}{2}\int_0^L \delta\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x &= \mathrm{EI}\int_0^L \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)\delta\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)\mathrm{d}x \\
&= \mathrm{EI}\int_0^L \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\frac{\mathrm{d}^2(\delta w)}{\mathrm{d}x^2}\mathrm{d}x \\
&= \left[\mathrm{EI}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\delta\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)\right]_0^L - \mathrm{EI}\int_0^L \frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\frac{\mathrm{d}(\delta w)}{\mathrm{d}x}\mathrm{d}x \\
&= \left[\mathrm{EI}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\delta\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)\right]_0^L - \left[\mathrm{EI}\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\delta w\right]_0^L \\
&\quad + \mathrm{EI}\int_0^L \frac{\mathrm{d}^4 w}{\mathrm{d}x^4}\delta w \mathrm{d}x
\end{aligned}
\tag{8.11}
$$

Substituting Equation (8.11) into (8.10) and rearranging, it is seen that:

$$
\begin{aligned}
\delta V = 0 = &\left[\mathrm{EI}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\delta\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)\right]_0^L - \left[\mathrm{EI}\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}\delta w\right]_0^L \\
&+ \int_0^L \left[\mathrm{EI}\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - q(x)\right]\delta w \mathrm{d}x
\end{aligned}
\tag{8.12}
$$

For this to be true, the following equation must be satisfied for the integral above to be zero:

$$
\mathrm{EI}\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q(x) \tag{8.13}
$$

This is obviously the governing equation for the bending of a beam under a lateral load. So, it is seen that if one considers a beam-type structure, uses beam assumptions, and uses proper stress-strain relations and strain-displacement relations, the result is the beam bending equation. However, it can be emphasized that if a nonclassical-shaped elastic structure were being analyzed, by using physical intuition, experience or some other reasoning to formulate stress-strain relations, and strain-displacement relations for the body, then through the Theorem of Minimum Potential Energy one can formulate the governing differential equations for the structure and load analogous to Equation (8.13). Incidentally, the resulting governing differential equations derived from the Theorem of Minimum Potential Energy are called the Euler-Lagrange equations.

Note also for Equation (8.12) to be true, each of the first two terms must be zero. Hence, at $x = 0$ and $x = L$ (at each end) either $\mathrm{EI}\left(\mathrm{d}^2 w/\mathrm{d}x^2\right) = -M_x = 0$ or $\left(\mathrm{d}w/\mathrm{d}x\right)$ must be specified (that is, its variation must be zero), also either $\mathrm{EI}\left(\mathrm{d}^3 w/\mathrm{d}x^3\right) = -V_x = 0$ or w must be specified. These are the natural boundary conditions. All of the classical boundary conditions, including simple supported, clamped and free edges are contained in the above "natural boundary conditions." This is a nice byproduct from using the variational approach for deriving governing equations for analyzing any elastic structure.

The above discussion shows that if in using The Theorem of Minimum Potential Energy one makes all of the assumptions of classical beam theory, the resulting Euler-Lagrange equation is the classical beam equation (8.13) and the natural boundary conditions given in (8.12) as discussed above.

Equally or more important the Theorem of Minimum Potential Energy provides a means to obtain an approximate solution to practical engineering problems by assuming good deflection functions which satisfy the boundary conditions. As the simplest example consider a beam simply supported at each end subjected to a uniform lateral load per unit length $q(x) = -q_0$, a constant.

Here, an example, assume a deflection which satisfies the boundary conditions for a beam simply supported at each end, where A is a constant to be determined.

$$w(x) = A \sin \frac{\pi x}{L} \tag{8.14}$$

This is not the exact solution, but should lead to a good approximation because (8.14) is a continuous single valued function which satisfies the boundary conditions of the problem.
Proceeding,

$$\begin{gathered} w' = A\frac{\pi}{L}\cos\frac{\pi x}{L}, \qquad w'' = -\frac{A\pi^2}{L^2}\sin\frac{\pi x}{L} \\ (w'')^2 = A^2\frac{\pi^4}{L^4}\sin^2\frac{\pi x}{L} \end{gathered} \tag{8.15}$$

Substituting (8.14) into (8.9) results in

$$\begin{aligned} V &= \frac{\mathrm{EI}}{2}\int_0^L \frac{A^2\pi^4}{L^4}\sin^2\frac{\pi x}{L}\mathrm{d}x - \int_0^L (-q_0)A\sin\frac{\pi x}{L}\mathrm{d}x \\ &= \frac{\mathrm{EI}}{2}\frac{A^2\pi^4}{L^4}\left(\frac{L}{2}\right) - (-q_0)A\frac{L}{\pi}\left[-\cos\frac{\pi x}{L}\right]_0^L \\ &= \frac{\pi^4}{4L^3}\mathrm{EI}\,A^2 + q_0 A\frac{L}{\pi}\left[-\cos\pi + 1\right] \end{aligned} \tag{8.16}$$

$$\delta V = 0 = \frac{\pi^4}{4L^3}\mathrm{EI}\,2A\delta A + \frac{2q_0 L}{\pi}\delta A = \delta A\left[\frac{\pi^4}{2L^3}\mathrm{EI}\,A + \frac{2q_0 L}{\pi}\right]$$

Therefore,

$$A = -\frac{4q_0 L^4}{\pi^5 \mathrm{EI}} = w(L/2) \tag{8.17}$$

The exact solution is

$$w(L/2) = -\frac{5}{384}\frac{q_0 L^4}{\mathrm{EI}} \tag{8.18}$$

The difference is seen to be 0.386%. So the Minimum Potential Energy solution is seen to be almost exact in determining the maximum deflection.

In determining maximum stresses the accuracy of the energy solution is less, because bending stresses are proportional to second derivatives of deflection. By taking derivatives the errors increase (conversely, integrating is an averaging process and errors decrease) so the stresses from the approximate solution differ more from the exact solution than do the deflections.

To continue this example for a one lamina composite beam, simply supported at each end, subjected to a constant uniform lateral load per unit length of $-q_0$, it is clear that the maximum stress occurs at $x = L/2$. From classical beam theory, the exact value of the maximum stress is

$$\sigma_{\max} = \sigma_x\left(\frac{L}{2}, \pm\frac{h}{2}\right) = \pm\frac{q_0 L^2}{8} \tag{8.19}$$

Likewise, for the Minimum Potential Energy solution, using (8.15)

$$\sigma_{\max} = \sigma_x\left(\frac{L}{2}, \pm\frac{h}{2}\right) = -\mathrm{EI}\, w''\left(\frac{L}{2}, \pm\frac{h}{2}\right) = \pm\frac{4q_0L^2}{\pi^3} \tag{8.20}$$

The difference between the two is 3.2%, so the energy solution is quite accurate for many applications.

If one wishes to increase the accuracy, instead of using (8.14) one could use

$$w(x) = \sum_{n=1}^{N} A_n \sin\frac{n\pi x}{L} \tag{8.21}$$

If N were chosen to be three, for example, the expression for $w(x)$ is given by $w(x) = A_1\sin\frac{\pi x}{L} + A_2\sin\frac{2\pi x}{L} + A_3\sin\frac{3\pi x}{L}$ and one would proceed as before, taking variations with respect to A_1, A_2 and A_3 which provides three algebraic equations for determining the three A_n. Of course as N increases, the accuracy of the solution increases until as N approaches infinity it is another form of the exact solution.

As a second example, examine the same beam, this time subjected to a concentrated load P at the mid-length, $x = L/2$. To obtain an exact solution, one must divide the beam into two parts, so that the load discontinuity can be accommodated, with the result that there are two fourth order differential equations and eight boundary conditions. Not so with the case of Minimum Potential Energy to obtain an approximate solution, as follows. Again assume (8.14) as the approximate deflection because it is single valued, continuous and satisfies the boundary conditions at the end of the beam. There,

$$\begin{aligned} V &= \frac{1}{2}\int_0^L \mathrm{EI}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x - Pw(L/2) \\ &= \frac{\pi^4}{4L^3}\mathrm{EI}\, A^2 - PA \end{aligned}$$

$$\delta V = 0 = \frac{\pi^4 \mathrm{EI}\, A\delta A}{2L^3} - P\delta A$$

$$\text{or, } A = \frac{2PL^3}{\pi^4 \mathrm{EI}} = w(L/2) = w_{\max}$$

Again, instead of (8.14) one could have chosen (8.21) as the trial function to use in solving this problem.

Thus, the Theorem of Minimum Potential Energy can be used easily for complicated laterally distributed loads, concentrated lateral loads, any boundary conditions, and/or variable or discontinuous beam thicknesses. One only needs to select a form of the lateral displacement such as the following examples.

Clamped Clamped Beam

$$w(x) = A\left[1 - \cos\frac{2\pi x}{L}\right] \tag{8.22}$$

Clamped-Simple Beam

$$w(x) = A\left[\mathrm{L}^3 x - 3Lx^3 + 2x^4\right] \tag{8.23}$$

Cantilevered Beam

$$w(x) = Ax^2 \tag{8.24}$$

8.4 The Buckling of Columns

In this case the strain energy is again given by Equation (8.8), where neglecting body forces F_i, the work done by surface tractions is given as follows:

$$\int_{S_T} T_i u_i \mathrm{d}S = -\int_0^L \left\{ P\left[\frac{\mathrm{d}u_0}{\mathrm{d}x} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \right] \right\} \mathrm{d}x.$$

This equation incorporates the more comprehensive theory employed in Chapter 6 to include buckling, and as discussed previously, to calculate buckling loads, $u_0 = 0$, because at incipient buckling the arc length of the buckled column is equal to the original length. Also, in the above, *P* is the tensile load, considered constant to make the problem linear. Therefore, for column buckling,

$$V = \frac{EI}{2}\int_0^L \left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)^2 \mathrm{d}x + \frac{P}{2}\int_0^L \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \mathrm{d}x. \tag{8.25}$$

Taking the variation of the potential energy, one obtains the following Euler-Lagrange equation analogous to Equation (8.13)

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - P\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0 \tag{8.26}$$

as well as the natural boundary conditions discussed previously. However, assuming a form of $w(x)$, which satisfies the boundary conditions for the column, which approximates the exact buckled shape will provide an approximation to the exact buckling load.

Consider a column simply supported at each end, if one uses (8.14) in (8.25) and takes variation of A, the result is:

$$V = \left[\frac{\pi^4 EI}{4L^3} + \frac{P\pi^2}{4L} \right] A^2$$

(8.27)

$$\delta V = \left[\frac{\pi^4 EI}{4L^3} - \frac{P\pi^2}{4L} \right] 2A\delta A = 0$$

so the bracket must equal zero, or

$$P = -\pi^2 \frac{EI}{L^2}.$$

It is seen that this is the exact buckling load, because the exact buckling mode (8.14) was utilized. Some other approximate displacement functions satisfying the boundary conditions would give an approximate buckling load. It can be proven that such an approximate buckling load will always be greater than the exact buckling load. However, as long as the assumed displacement satisfies the boundary conditions, the error is never more than a very few percent of the exact value.

8.5 Vibration of Beams

The energy principle to utilize in dynamic analysis is Hamilton's Principle which employs the functional

$$I = \int_{t_1}^{t_2} (T - V)\mathrm{d}t \,. \tag{8.28}$$

Hamilton's Principle states that in a conservative system

$$\delta I = 0 \,. \tag{8.29}$$

In the above, the potential energy, V, is given by Equation (8.1), and T is the kinetic energy of the body. In a beam undergoing flexural vibration, the kinetic energy would be

$$T = \frac{1}{2}\int_0^L \rho_m A\left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x \tag{8.30}$$

where ρ_m is the *mass* density of the material, A is the beam cross-sectional area, and $\partial w/\partial t$ is the velocity of the beam.

Using Hamilton's Principle in the same way that was done before for Minimum Potential Energy, the resulting Euler-Lagrange equation is

$$EI\frac{\partial^4 w}{\partial x^4} + \rho_m A\frac{\partial^2 w}{\partial t^2} = 0 \tag{8.31}$$

which is identical to Equation (7.3). Also resulting are the natural boundary conditions, discussed previously.

Considering a beam simply supported at each end, if Equation (8.14) is modified to include a harmonic motion with time, such as

$$w(x,t) = C\sin\frac{n\pi x}{L}\cos\omega_n t$$

where C is a constant.
The result is an Euler-Lagrange equation of

$$\omega_n = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{\rho_m A}} \tag{8.32}$$

which is the exact solution for the natural circular frequency, ω_n, in radians/unit time [see Equation (7.6)] because the exact mode shape was assumed. Again, if the assumed displacement function is approximate, then approximate natural frequencies will be obtained; are higher than the exact frequencies, but the error will be at most a few percent. In any case the natural frequency, f (in Hz), is found by $\omega_n/2\pi$.

Note that in assuming mode shape functions in both buckling and vibration problems (eigenvalue problems), the closer the assumed approximate function is to the exact mode shape, the lower the resulting eigenvalue will be, and of course it will be closer to the exact eigenvalue, since the exact eigenvalue is always lower than any approximated value.

8.6 Minimum Potential Energy for Rectangular Isotropic Plates

The strain energy density function, W, for a three dimensional solid in rectangular coordinates is given by Equation (8.4). The assumptions associated with the classical plate theory of Chapter 2 are employed to modify (8.4) for a rectangular plate. If transverse shear deformation is neglected, then $\varepsilon_{xz} = \varepsilon_{yz} = 0$. If there is no plate thickening, then $\varepsilon_z = 0$. From Equations (1.9), (1.10), and (1.12), stresses are written in terms of strains, such that for the classical plate,

$$\sigma_x = \frac{E}{(1-\nu^2)}[\varepsilon_x + \nu\varepsilon_y]; \quad \sigma_y = \frac{E}{(1-\nu^2)}[\varepsilon_y + \nu\varepsilon_x]; \quad \sigma_{xy} = \frac{E}{(1-\nu^2)}\varepsilon_{xy}. \tag{8.33}$$

Therefore, (8.4) becomes

$$W = \frac{E\varepsilon_x}{2(1-\nu^2)}(\varepsilon_x + \nu\varepsilon_y) + \frac{E\varepsilon_y}{2(1-\nu^2)}(\varepsilon_y + \nu\varepsilon_x) + \frac{E}{(1+\nu)}\varepsilon_{xy}^2. \tag{8.34}$$

If the plate is subjected to bending and stretching, the deflection functions are given by Equations (2.24) through (2.28). Substituting these into (8.34) results in the following:

$$\begin{aligned} W = {} & \frac{1}{2}\frac{E}{(1-\nu^2)}\left\{\left(\frac{\partial u_0}{\partial x}\right)^2 + \left(\frac{\partial v_0}{\partial y}\right)^2 + 2\nu\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial v_0}{\partial y}\right) + \frac{1+\nu}{2}\left[\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right]^2\right\} \\ & + \frac{Ez^2}{2(1-\nu^2)}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right)\right] \\ & + \frac{Ez^2}{(1+\nu)}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2. \end{aligned} \tag{8.35}$$

From this the strain energy $U(=\int_R W \mathrm{d}R)$ is found.

$$\begin{aligned} U = {} & \frac{K}{2}\int_0^a\int_0^b\left\{\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right)^2 - 2(1-\nu)\frac{\partial u_0}{\partial x}\frac{\partial v_0}{\partial y} + \frac{1-\nu}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right)^2\right\}\mathrm{d}x\mathrm{d}y \\ & + \frac{D}{2}\int_0^a\int_0^b\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\}\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{8.36}$$

It is seen that the first term is the extensional or in-plane strain energy of the plate, and the second is the bending strain energy of the plate. In the latter, it is seen that the first term is proportional to the square of the average plate curvature, while the second term is known as the Gaussian curvature.

For the plate the total work term due to surface traction is seen to be

$$\int_{S_T} T_i u_i \,\mathrm{d}S = \int_0^1 \int_0^b p(x,y)\, w(x,y)\, \mathrm{d}x\mathrm{d}y - \int_0^a \int_0^b \left\{ N_x \left[\frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right] \right.$$
$$\left. + N_y \left[\frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \right] + N_{xy} \left[\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \right] \right\} \mathrm{d}x\mathrm{d}y. \tag{8.37}$$

Hence, in (8.36) and (8.37) if one considers a plate subjected only to a lateral load $p(x, y)$, one assumes $u_0 = v_0 = N_x = N_y = N_{xy} = 0$. If one is considering in-plane loads only (except for buckling) assume $w(x, y) = p(x, y) = 0$. If one is looking for buckling loads, assume $p(x,y) = u_0 = v_0 = 0$. The rationale for all of this has been discussed previously.

8.7 The Buckling of an Isotropic Plate Under a Uniaxial Load, Simply Supported on Three Sides, and Free on an Unloaded Edge

The most beneficial use of the Minimum Potential Energy Theorem occurs when one cannot formulate a suitable set of governing differential equations, and/or when one cannot ascertain a consistent set of boundary conditions. In that case one can make a reasonable assumption of the displacements, and then solves for an approximate solution using the Theorem of Minimum Potential Energy. This is illustrated in the following example.

Consider the plate shown below in Figure 8.2. The governing differential equation for this problem is obtained from Equation (6.7).

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{N_x}{D}\frac{\partial^2 w}{\partial x^2}. \tag{8.38}$$

To solve for the buckling load directly, a Levy type solution may be assumed:

$$w(x,y) = \sum_{m=1}^{\infty} \psi_m(y) \sin\frac{m\pi x}{a}. \tag{8.39}$$

Substituting (8.39) into (8.38) results in the following ordinary differential equation to solve:

$$\lambda_m^4 \psi - 2\lambda_m^2 \psi'' + \psi^{IV} = -\frac{N_x}{D}\lambda_m^2 \psi \tag{8.40}$$

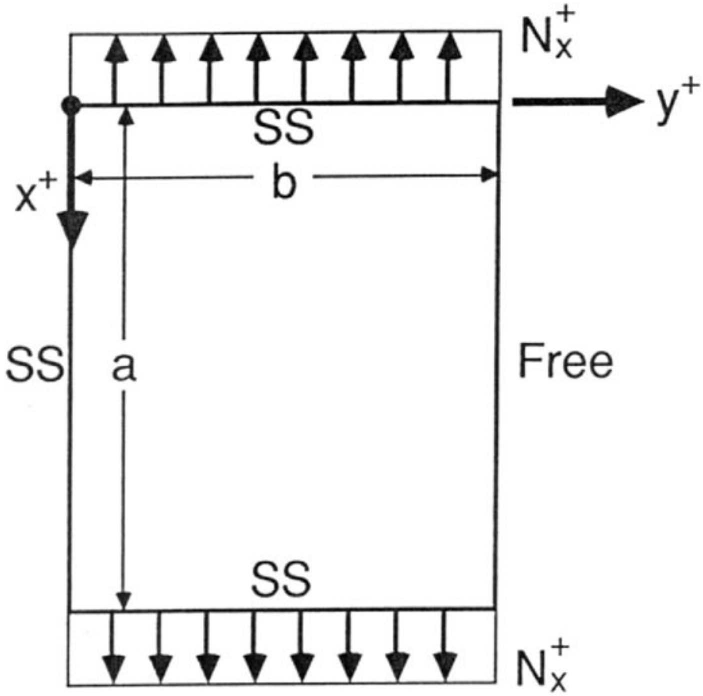


Figure 8.2. Plate studied in Section 8.7.

where

$$\lambda_m = \frac{m\pi}{a}, \quad (\ )'' = \frac{\mathrm{d}^2(\ )}{\mathrm{d}y^2}, \quad \text{and} \quad (\ )^{IV} = \frac{d^4(\ )}{\mathrm{d}y^4}$$

Letting $\overline{N}_x = -N_x$, Equation (8.40) can be solved with the result that

$$\psi_m(y) = A\cosh\alpha\, y + B\sinh\alpha\, y + C\cos\beta\, y + E\sin\beta\, y \tag{8.41}$$

where

$$\alpha = \left[\lambda_m^2 + \lambda_m\sqrt{\frac{\overline{N}_{xm}}{D}}\right]^{1/2}$$

$$\beta = \left[-\lambda_m^2 + \lambda_m\sqrt{\frac{\overline{N}_{xm}}{D}}\right]^{1/2}.$$

The boundary conditions on the $y = 0$ and b edges are

$$\begin{aligned} &w(x,0) = 0 \rightarrow \psi(0) = 0 \\ &M_y(x,0) = 0 \rightarrow \psi''(0) = 0 \\ &M_y(x,b) = 0 \rightarrow \psi''(b) - \nu\lambda_m^2\psi(b) = 0 \\ &V(x,b) = 0 \rightarrow \psi'''(b) - (2-\nu)\lambda_m^3\psi'(b) = 0. \end{aligned} \tag{8.42}$$

It is clear that the first two boundary conditions require that $A = C = 0$. Satisfying the other two boundary conditions results in the following relationship for the eigenvalues (i.e., the buckling load $\overline{N}_x = -N_x$).

$$-\beta \tanh\alpha\, b[\alpha^2 - \nu\lambda_m^2]^2 + \alpha \tan\beta\, b[\beta^2 + \nu\lambda_m^2]^2 = 0. \tag{8.43}$$

Thus, knowing the plate geometry and the material properties, one can solve for the buckling loads for each value of m. It can be shown that the minimum buckling load will occur for $m = 1$, thus a one-half sine wave in the longitudinal direction. However, note the complexity both in obtaining Equation (8.43), and then using that equation to obtain the buckling load, compared to the relative simplicity of Section 6.4 for solving the simpler problem of the plate completely simple supported on all four edges. The solutions of this problem have been catalogued in Reference 6.1 and are given below:

$$N_x = -\frac{k\pi^2 D}{b^2} \quad \text{and} \quad \sigma_{\text{cr}} = -\frac{k\pi^2 E}{12(1-\nu^2)}\left(\frac{h}{b}\right)^2.$$

For $\nu = 0.25$

a/b	0.50	1.0	2.0	3.0	4.0	5.0
k	4.40	1.44	0.698	0.564	0.516	0.506

Now to solve the same problem using Minimum Potential Energy. However, before doing so a brief discussion regarding boundary conditions is in order. They can be divided into two categories: geometric and stress. Geometric boundary conditions involve specifications on the displacement function and the first derivative, such as specifying the lateral displacement w or the slope at the boundary, $\partial w/\partial x$ or $\partial w/\partial y$, stress boundary conditions involve the specifications of the second and third derivative of the displacement function, such as the stress couples, M_x, M_y, M_{xy}, or the transverse shear resultants Q_x, Q_y, or the effective transverse shear resultants V_x, V_y, discussed in Chapter 2.

In using the Minimum Potential Energy Theorem, one must choose a deflection function that *at least* satisfies the geometric boundary conditions specified on the boundaries. This suitable function will give a reasonable approximate solution. Better yet, by assuming a deflection function that satisfies all specified boundary conditions, one can achieve a very good approximate solution. If one could choose a deflection function that satisfies all boundary conditions and the governing differential equation for the problem also, that is the exact solution! Finally, if one chose a deflection function that did not satisfy even the geometric boundary conditions, the solution would be inaccurate because in effect the solution would not be for the problem to be solved, but for some other problem for which the assumed deflection does satisfy the geometric boundary conditions.

In this example, the following function is assumed for the lateral deflection:

$$w(x,y) = Ay\sin\frac{\pi x}{a} \tag{8.44}$$

This satisfies all boundary conditions on the x = 0, a edges. It satisfies the geometric boundary condition that $w(x,0) = 0$, but does not satisfy the stress boundary conditions that $M(x,0) = M(x,b) = V(x,b) = 0$. Substituting Equation (8.44) and its derivatives into Equations (8.1) using (8.36) and (8.37), where of course $N_y = N_{xy} = p(x,y) = 0$ produces

$$\begin{aligned} V &= \frac{D}{2}\int_0^a\int_0^b\left\{\left[-Ay\frac{\pi^2}{a^2}\sin\frac{\pi x}{a}\right]^2 + 2(1-\nu)\left[-A\frac{\pi}{a}\cos\frac{\pi x}{a}\right]^2\right\}\mathrm{d}x\mathrm{d}y \\ &+\frac{N_x}{2}\int_0^a\int_0^b A^2y^2\frac{\pi^2}{a^2}\cos^2\frac{\pi x}{a}\,\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{8.45}$$

Integrating Equation (8.45) gives

$$V = A^2D\left[\frac{\pi^4}{a^3}\frac{b^3}{3} + 2(1-\nu)\frac{\pi^2 b}{a}\right] + N_xA^2\frac{\pi^2b^3}{3a}.$$

Setting $\delta V = 0$, where the only variable with which to take a variation is A, produces the requirement that

$$N_x = -\left[\frac{\pi^2 D}{a^2} + \frac{6(1-\nu)D}{b^2}\right]. \tag{8.46}$$

To compare this approximate result with the exact solution shown previously, let $a/b = 1$, and $\nu = 0.25$. From Equation (8.46)

$$N_{x_{\mathrm{cr}}} = -1.456\frac{\pi^2 D}{b^2}. \tag{8.47}$$

In the exact solution, the coefficient is 1.440. Hence, the difference between the approximate solution and the exact solution is approximately 1%, yet the three stress boundary conditions on the y = constant edges were not satisfied.

8.8 Functions for Displacements in Using Minimum Potential Energy for Solving Beam, Column, and Plate Problems

In the use of Minimum Potential Energy methods to solve beam, column, and plate problems, one usually needs to assume an expression for the lateral deflection $w(x)$ for the beam or column, and $w(x, y)$ for the plate. These must be single valued, continuous functions that satisfy all the boundary conditions, or at least the geometric ones. Below are a few functions useful in the solutions of beam and column problems.

Simple-simple

$$w(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \tag{8.48}$$

Simple-free

$$w(x) = Ax \tag{8.49}$$

Clamped-clamped

$$w(x) = A\left[1 - \cos\frac{2m\pi x}{a}\right] \tag{8.50}$$

Clamped-free

$$w(x) = Ax^2 \tag{8.51}$$

Clamped-simple

$$w(x) = A[L^3 x - 3Lx^3 + 2x^4] \tag{8.52}$$

Free-free

$$w = A. \tag{8.53}$$

In the case of a plate with varied boundary conditions, let $w(x, y) = f(x)g(x)$ where for $f(x)$ and $g(y)$ use the appropriate beam functions above. For example, consider a plate clamped on edges $y = 0$ and $y = b$, and clamped at $x = 0$ and simply supported at $x = a$. Assume the function:

$$w(x,y) = A_m[L^3x - 3Lx^3 + 2x^4]\left[1 - \cos\frac{2m\pi y}{b}\right]. \tag{8.54}$$

Keep in mind none of the above functions are unique, and thus the engineer may use his ingenuity to conceive functions best for the solution of that particular problem. For instance, suppose a plate had one edge simply supported at $y = 0$, $0 \le x \le a/2$, and clamped from $a/2 \le x \le a$. No analytical solution could be obtained but an approximate solution using energy methods is always attainable.

Perhaps the most complete and useful tabulation of functions, their derivatives and their integrals, to use in energy methods are those of Warburton [8.3] and Young and Felgar [3.1, 3.2].

8.9 References

8.1. Reissner, E. (1950) On a Variational Theorem in Elasticity, *J. Math. Phys.*, Vol. 29, pg. 90.

8.2. Mura, T. and Koya, T. (1992) *Variational Methods in Mechanics*, Oxford University Press, March.

8.3. Warburton, G. (1968) The Vibration of Rectangular Plates, *Proceedings of the Institute of Mechanical Engineers*, pp. 371-384.

8.10 Problems

8.1. Consider a steel plate ($E = 30\times10^6$ psi, $\nu = 0.25, \sigma_y = 30{,}000$ psi) used as a portion of a bulkhead on a ship. The bulkhead is 60″ long and 30″ wide subjected to an in-plane compressive load in the longer direction. What thickness must the plate be to have a buckling stress equal to the yield stress if:
(a) the plate is simply supported on all four edges?
(b) the plate is simply supported on three edges and free on one unloaded edge?

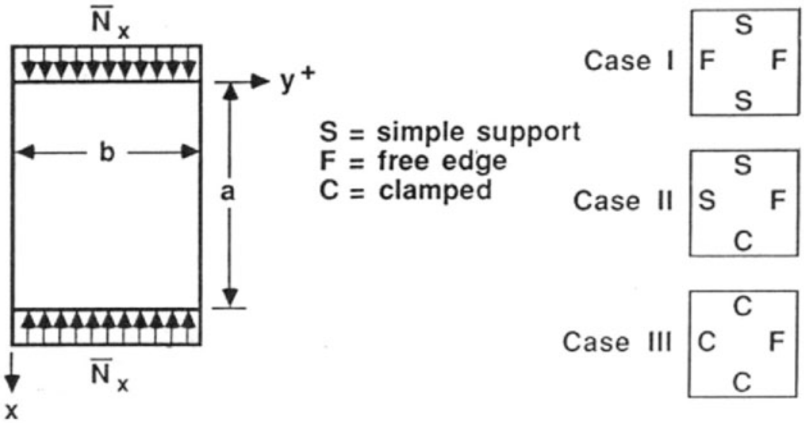
8.2. Given a column of width *b*, height *h*, and length *L*, simply supported at each end, use the principle of Minimum Potential Energy to determine the buckling load, if one assumes the deflection to be

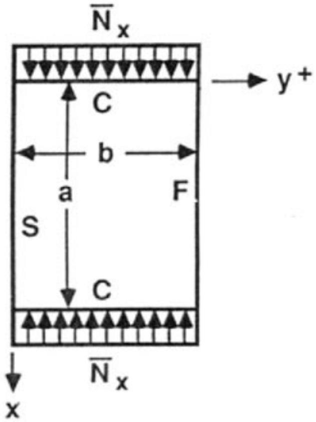
(a) $w(x) = A\dfrac{x}{L}(L-x)$

(b) $w(x) = \dfrac{A}{L^3}[2Lx^3 - x^4 - L^3x]$

where in each case A is an amplitude.
Do the deflections assumed above satisfy the geometric boundary conditions? Do they satisfy the stress boundary conditions?

8.3. Consider the plates below, each subjected to a uniform axial compressive load per inch of width, $\overline{N}_x = -N_x$ (lbs./in.) in the x direction. Determine a suitable deflection function $w(x, y)$ for each case for subsequent use in the Principle of Minimum Potential Energy to determine the critical load $\overline{N}_x$.





8.4. For an end plate in a support structure with the following boundary conditions, use the Principle of Minimum Potential Energy to determine the buckling load, if one assumes the deflection function to be $w = A[1-\cos(2\pi x/a)]$, where A is the unknown amplitude.

8.5. Consider a rectangular plate of $0 \le x \le a$, $0 \le y \le b$, $-h/2 \le z \le h/2$. If the lateral deflection $w(x, y)$ is assumed to be in a separable form $w = f(x)g(y)$, and if $w = 0$ on all boundaries, determine the amount of strain energy due to the terms comprising the Gaussian curvature. See (8.36).

8.6. The base of a missile launch platform consists in part of vertical rectangular plates of height a, and width b, where $a > b$. They are tied into the foundation below and the platform above such that those edges are considered clamped.

However, on their vertical edges they are tied into I-beams, such that those edges can only be considered simply supported. Using the Theory of Minimum Potential Energy, derive the equation for the buckling load per inch of edge distance, $N_{x_{\mathrm{cr}}}$, for these plates, using a suitable deflection function, so that the plates can be designed to resist buckling.

8.7. An alternative to the design of Problem 8.6 would be to 'beef up' the vertical support beams such that the plate members can be considered to have their vertical edges clamped. Thus the plates have all four edges clamped. Employing a suitable deflection function, use the Theorem of Minimum Potential Energy to determine an expression for the critical buckling load per unit edge distance, $N_{x_{\mathrm{cr}}}$, to use in designing the plates. Is the plate with all edges clamped thicker or thinner than the one with the sides simply supported in Problem 8.6, to have the same buckling load?

8.8. The legs of a water tower consist of three columns of length a, constant flexural stiffness EI, simply supported at one end and clamped at the other end. Using the Theorem of Minimum Potential Energy, and a suitable function for the lateral deflection, calculate the buckling load P_{cr} for each leg, in order that they may be properly designed.

8.9. Consider a beam of length L, and constant cross-section, i.e., EI is a constant. The beam is subjected to a load $q(x) = a + c(x/L)$, (lbs./in.) applied laterally where a and c are constants. The beam is simply supported on both ends. Using Minimum Potential Energy, and assuming $w(x) = B\sin(\pi x/L)$, determine the maximum deflection, w, and the maximum bending stress, σ_x. Consider the beam to be of unit width, i.e., $b = 1$.

8.10. A beam of length L, and constant cross-section (EI = constant) is subjected to a lateral load $q(x) = (q_0 x)/L$, where q_0 is a constant, and is simply supported at each end. Using Minimum Potential Energy, and assuming $w(x) = A\sin(\pi x/L)$, where A is a constant to be determined, determine the maximum deflection, w, and the maximum stress, σ_x, in the beam.

8.11. Consider the beam of Section 8.3 to be simply supported at each end and subjected to a uniform lateral load q_0 (lbs./in.). Assuming the deflection to be $w(x) = A\sin(\pi x/L)$, use the Principle of Minimum Potential Energy to determine A.

8.12. Consider a beam-column simply supported at one end and clamped at the other. Using the Theorem of Minimum Potential Energy, and assuming an admissible form for the lateral deflection, $w(x)$, calculate the in-plane load, P_{cr} (lbs.), to buckle the column.

8.13. Consider a beam of stiffness EI, length L, width b, height h, simply supported at each end, subjected to a uniform lateral load, q_0 (lbs./in.). Use Minimum Potential Energy, employing a deflection function

$$w(x) = \sum_{n=1}^{N} A_n \sin\frac{n\pi x}{L}$$

where N = 3, to determine the maximum deflection and maximum stress. Compare the answer with the exact solution.

8.14. Consider a column of length L, clamped at one end and simply supported at the other end. Using a buckling mode shape of

$$w(x) = A[L^3 x - 3Lx^3 + 2x^4]$$

where A is the buckle amplitude. Use Minimum Potential Energy to determine the axial critical buckling load, P_{cr}.

8.15. Consider a beam of constant flexural stiffness EI, of length L, clamped at each end. Using Hamilton's Principle, and an assumed deflection of

$$w(x,t) = A[1 - \cos(2\pi x / L)]\sin\omega_n t\,,$$

determine the fundamental natural frequency, and compare it with the exact solution.